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HOLOMORPHIC ETA QUOTIENTS OF WEIGHT 1/2

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Abstract. We give a short proof of Zagier's conjecture / Mersmann's theorem which states that each holomorphic eta quotient of weight 1/2 is an integral rescaling of some eta quotient from Zagier's list of fourteen primitive holomorphic eta quotients. In particular, given any holomorphic eta quotient f of weight 1/2, this result enables us to provide a closed-form expression for the coefficient of q^n in the q-series expansion of f, for all n. We also demonstrate another application of the above theorem in extending the levels of the simple (resp. irreducible) holomorphic eta quotients.

1. Introduction

The Dedekind eta function is defined by the infinite product:

$$\eta(z) := e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty}(1-e^{2\pi i n z}) \tag{1.1}$$

for all $z \in \mathfrak{H}$, where $\mathfrak{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$. Eta is a holomorphic function on $\mathfrak{H}$ with no zeros. This function comes up naturally in many areas of Mathematics (see the Introduction in [2] for a brief overview of them). The function η is a modular form of weight 1/2 with a multiplier system on $\mathrm{SL}_2(\mathbb{Z})$ (see [10]). An eta quotient f is a finite product of the form

$$\prod \eta_d^{X_d}, \tag{1.2}$$

where $d \in \mathbb{N}$, η_d is the *rescaling* of η by d, defined by

$$\eta_d(z) := \eta(dz) \quad \text{for all } z \in \mathfrak{H} \tag{1.3}$$

and the *exponents* $X_d \in \mathbb{Z}$. Eta quotients naturally inherit modularity from η: The eta quotient f in (1.2) transforms like a modular form of weight $\frac{1}{2}\sum_d X_d$ with a multiplier system on suitable congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$: The largest among these subgroups is

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\}, \tag{1.4}$$

where

$$N := \mathrm{lcm}\{d \in \mathbb{N} \,|\, X_d \neq 0\}. \tag{1.5}$$

We call N the *level* of f. Since η is non-zero on $\mathfrak{H}$, the eta quotient f is holomorphic if and only if f does not have any pole at the cusps of $\Gamma_0(N)$.

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We call an eta quotient f *primitive* if there does not exist any other eta quotient h and any $\nu \in \mathbb{N}$ such that $f(z) = h(\nu z)$ for all $z \in \mathfrak{H}$.

An *eta quotient on* $\Gamma_0(M)$ is an eta quotient whose level divides M. Let f, g and h be nonconstant holomorphic eta quotients on $\Gamma_0(M)$ such that $f = g \times h$. Then we say that f is *factorizable on* $\Gamma_0(M)$. We call a holomorphic eta quotient f of level N *quasi-irreducible* (resp. *irreducible*), if it is not factorizable on $\Gamma_0(N)$ (resp. on $\Gamma_0(M)$ for all multiples M of N). Here, it is worth mentioning that the notions of irreducibility and quasi-irreducibility of holomorphic eta quotients are conjecturally equivalent (see [2]). We say that a holomorphic eta quotient is *simple* if it is both primitive and quasi-irreducible.

2. Zagier's list

Zagier observed that every holomorphic eta quotient of weight $1/2$ seems to originate through integral rescalings of only a small number of primitive eta quotients. He gave an explicit list (see [16] or Theorem 1 below) of fourteen primitive holomorphic eta quotients of weight $1/2$ and conjectured that the list is complete. This conjecture was established by his student Mersmann in an excellent *Diplomarbeit* [13]. In his thesis, Mersmann also proved a more general conjecture of Zagier which asserts that: *There are only finitely many simple holomorphic eta quotients of a given weight.* I provided a much simplified proof of this conjecture in [4]. Furthermore in [3], I showed that the finiteness also holds if we replace the word "weight" with "level" in the above conjecture. In particular, since $1/2$ is the smallest possible weight of any holomorphic eta quotient, no such eta quotient of weight $1/2$ is a product of two holomorphic eta quotients other than that of 1 and itself. Thus, Mersmann's classification of holomorphic eta quotients of weight $1/2$ is only a special case of the last conjecture worked out in complete details:

Theorem 1 (Classification of Holomorphic Eta Quotients of Weight $1/2$)**.** *Each holomorphic eta quotient of weight* $1/2$ *is a rescaling of one of the following eta quotients by a positive integer:*

$$\eta, \frac{\eta^2}{\eta_2}, \frac{\eta_2^2}{\eta}, \frac{\eta_2^3}{\eta\eta_4}, \frac{\eta_2^5}{\eta^2\eta_4^2}, \frac{\eta\eta_4}{\eta_2}, \frac{\eta\eta_6^2}{\eta_2\eta_3}, \frac{\eta^2\eta_6}{\eta_2\eta_3}, \frac{\eta_2^2\eta_3}{\eta\eta_6}, \frac{\eta_2\eta_3^2}{\eta\eta_6}$$

$$\frac{\eta_2^2\eta_3\eta_{12}}{\eta\eta_4\eta_6}, \frac{\eta_2^5\eta_3\eta_{12}}{\eta^2\eta_4^2\eta_6^2}, \frac{\eta\eta_4\eta_6^2}{\eta_2\eta_3\eta_{12}}, \frac{\eta\eta_4\eta_6^5}{\eta_2^2\eta_3^2\eta_{12}^2}.$$

Though Mersmann's proof of the above theorem is indeed ingenious (see [13]), but Köhler at p. 117 in [10] also complains about its length and lack of lucidity. So, after briefly discussing some applications of Theorem 1 in the next two sections, we shall present a shorter and simpler proof of the theorem. Except in a few places (for example, see the proof of Lemma 4), we shall closely follow the basic ideas in Mersmann's proof.

3. q-series expansions of the eta quotients in Zagier's list

Theorem 1 implies that in order to obtain the q-series expansion of any holomorphic eta quotient of weight 1/2, it suffices only to know the q-series expansions of the eta quotients in Zagier's list. Recall that the Jacobi triple product identity states:

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}y)(1+x^{2n-1}y^{-1}) = \sum_{n=-\infty}^{\infty} x^{n^2}y^n \tag{3.1}$$

for all $x, y \in \mathbb{C}$ such that $|x| < 1$ and $y \neq 0$ (see Theorem 352 in [8]). Suitable substitutions in the above identity shows that each of the eta quotients in Zagier's list has a theta series representation with fractional exponents (for a list of the required substitutions, see Table 1 below or Remark 4.9 in [6]). These theta series representations of the eta quotients in Zagier's list are not much of a surprise since Serre-Stark theorem [15] implies that all suitable integral rescalings of holomorphic eta quotients of weight 1/2 are linear combinations of theta series of the form

$$\sum_{n\in\mathbb{Z}} \psi(n)q^{tn^2}, \tag{3.2}$$

where ψ is an even Dirichlet character, $t \in \mathbb{N}$ and

$$q^r = q^r(z) := e^{2\pi i r z} \text{ for all } r.$$

Table 1. Zagier's list via substitutions in the Jacobi triple product

$x \backslash y$	$q^{1/2}$	$iq^{1/2}$	$-q^{1/2}$	$-iq^{1/2}$	$\omega q^{1/2}$	$i\omega q^{1/2}$	$-q^2$	$-\omega q^2$
$q^{1/2}$	$\frac{\eta_2^2}{\eta}$				$\frac{\eta^2\eta_6}{\eta_2\eta_3}$			
$iq^{1/2}$		$\frac{\eta\eta_4}{\eta_2}$				$\frac{\eta_2^5\eta_3\eta_{12}}{\eta^2\eta_4^2\eta_6^2}$		
q							$\frac{\eta^2}{\eta_2}$	$\frac{\eta_2^2\eta_3}{\eta\eta_6}$
$-q$							$\frac{\eta_2^5}{\eta^2\eta_4^2}$	$\frac{\eta\eta_4\eta_6^2}{\eta_2\eta_3\eta_{12}}$
$q^{3/2}$	$\frac{\eta_2\eta_3^2}{\eta\eta_6}$		η					
$iq^{3/2}$		$\frac{\eta_2^3}{\eta\eta_4}$		$\frac{\eta\eta_4\eta_6^5}{\eta_2^2\eta_3^2\eta_{12}^2}$				
q^3							$\frac{\eta\eta_6^2}{\eta_2\eta_3}$	
$-q^3$							$\frac{\eta_2^2\eta_3\eta_{12}}{\eta\eta_4\eta_6}$	

In the above table, ω denotes a primitive cube root of unity. Making the substitutions in (3.1) as detailed in the Table 1, we see that for each eta quotient f in Zagier's list, there exists a $t|24$ and a sequence $\{a_n\}_n$ of complex numbers such that

$$f(z) = \sum_n a_n q^{tn^2/24}. \tag{3.3}$$

(see also [1], [16], [17], [18] or Chapter 8 in [10]). In particular, identifying the eta quotients in Zagier's list whose corresponding sequences $\{a_n\}_n$ define Dirichlet characters, we obtain an elementary proof of Theorem 1.1.1 in [11].

Also, from Table 1, (3.1) and (1.1), we see that the translation $z \mapsto z + \frac{1}{2}$ or equivalently the *sign transform* $q \mapsto -q$ induces an involution of Zagier's list of eta quotients up to multiplication by a 48-th root of unity (see [10], [13], [17] and [18]).

4. Extending the levels of simple holomorphic eta quotients

Recall that a simple holomorphic eta quotient is both primitive and quasi-irreducible. In particular, since primitivity and simplicity are synonymous for holomorphic eta quotients of weight $1/2$, Zagier's list supplies us with examples of simple holomorphic eta quotients of levels $1, 2, 4, 6$ and 12. Also, Corollary 2 in [2] implies that there are simple holomorphic eta quotients of all prime levels. Thus, we see that there exist simple holomorphic eta quotients of all levels less than 8. It is easy to check that, any quasi-irreducible holomorphic eta quotient of level 8 is a rescaling of some eta quotient of level 1,2 or 4. In other words, there does not exist any simple holomorphic eta quotient of level 8. It is still an open problem to classify the levels of which simple holomorphic eta quotients exist. However, Theorem 1 implies that given a simple (resp. irreducible) holomorphic eta quotient f of a suitable level, we can construct up to thirteen new simple (resp. irreducible) holomorphic eta quotients of the same weight as of f but of higher levels:

Corollary 1. *Let there be a simple* (*resp. irreducible*) *holomorphic eta quotient of an odd level N. Then there are at least two simple* (*resp. irreducible*) *holomorphic eta quotients of level $2N$ and and at least three simple* (*resp. irreducible*) *holomorphic eta quotients of level $4N$. Moreover, if $3 \nmid N$, then there are also four simple* (*resp. irreducible*) *holomorphic eta quotients of levels $6N$ and four simple* (*resp. irreducible*) *holomorphic eta quotients of $12N$.*

Proof. Let f be a holomorphic eta quotient of level N. Let g be a holomorphic eta quotient of weight $1/2$ from Zagier's list (see Theorem 1) such that N is coprime to the level of g. Since g is primitive, it follows that f is primitive if and only if so is

$$f \circledast g := \prod_{d|M} \prod_{d'|N} \eta_{dd'}^{X_d Y_{d'}}, \tag{4.1}$$

where $f = \prod_{d|M} \eta^{X_d}$ and $g = \prod_{d'|N} \eta^{Y_{d'}}$. Again, since g is of weight $1/2$, from Lemma 2 and Corollary 5 in [2], it follows that $f \circledast g$ is quasi-irreducible (resp. irreducible) if and only if so is f. □

Section 6 in [3] furnishes several examples of irreducible holomorphic eta quotients. In particular, if N is cubefree (i. e., if N is not divisible by the cube of any integer except 1), then the holomorphic eta quotient of level N constructed in the proof of Theorem 3 in [3] is in particular, simple (see also Theorem 6.2 and Corollary 6.3 in [6]). We shall also supply an infinite family of simple holomorphic eta quotients in [5] whose levels are not cubefree.

5. Notations and the basic facts

By $\mathbb{N}$ we denote the set of positive integers. For $N \in \mathbb{N}$, by $\mathcal{D}_N$ we denote the set of divisors of N. For $X \in \mathbb{Z}^{\mathcal{D}_N}$, we define the eta quotient η^X by

$$\eta^X := \prod_{d \in \mathcal{D}_N} \eta_d^{X_d}, \tag{5.1}$$

where X_d is the value of X at $d \in \mathcal{D}_N$ whereas η_d denotes the rescaling of η by d. Clearly, the level of η^X divides N. In other words, η^X transforms like a modular form on $\Gamma_0(N)$. We define the summatory function $\sigma : \mathbb{Z}^{\mathcal{D}_N} \to \mathbb{Z}$ by

$$\sigma(X) := \sum_{d \in \mathcal{D}_N} X_d. \tag{5.2}$$

Since η is of weight $1/2$, the weight of η^X is $\sigma(X)/2$ for all $X \in \mathbb{Z}^{\mathcal{D}_N}$.

An *eta quotient on* $\Gamma_0(N)$ is an eta quotient whose level divides N. We recall that such an eta quotient f is holomorphic if it does not have any poles at the cusps of $\Gamma_0(N)$. Under the action of $\Gamma_0(N)$ on $\mathbb{P}^1(\mathbb{Q})$ by Möbius transformation, for $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$, we have

$$[a : b] \sim_{\Gamma_0(N)} [a' : \gcd(N, b)] \tag{5.3}$$

for some $a' \in \mathbb{Z}$ which is coprime to $\gcd(N, b)$ (see [7]). We identify $\mathbb{P}^1(\mathbb{Q})$ with $\mathbb{Q} \cup \{\infty\}$ via the canonical bijection that maps $[\alpha : \lambda]$ to α/λ if $\lambda \neq 0$ and to ∞ if $\lambda = 0$. For $s \in \mathbb{Q} \cup \{\infty\}$ and a weakly holomorphic modular form f on $\Gamma_0(N)$, the order of f at the cusp s of $\Gamma_0(N)$ is the exponent of q^{1/w_s} occurring with the first nonzero coefficient in the q-expansion of f at the cusp s, where w_s is the width of the cusp s (see [7], [14]). The following is a minimal set of representatives of the cusps of $\Gamma_0(N)$ (see [7], [12]):

$$\mathcal{S}_N := \Big\{ \frac{a}{t} \in \mathbb{Q} \;\Big|\; t \in \mathcal{D}_N,\, a \in \mathbb{Z},\, \gcd(a, t) = 1 \Big\} / \sim, \tag{5.4}$$

where $\dfrac{a}{t} \sim \dfrac{b}{t}$ if and only if $a \equiv b \pmod{\gcd(t, N/t)}$. For $d \in \mathcal{D}_N$ and for $s = \dfrac{a}{t} \in \mathcal{S}_N$ with $\gcd(a, t) = 1$, we have

$$\operatorname{ord}_s(\eta_d; \Gamma_0(N)) = \frac{N \cdot \gcd(d, t)^2}{24 \cdot d \cdot \gcd(t^2, N)} \in \frac{1}{24}\mathbb{N} \tag{5.5}$$

(see [12]). It is easy to check the above inclusion when N is a prime power. The case for general N follows by multiplicativity (see (5.8), (5.10) and (5.12)). It follows that for all $X \in \mathbb{Z}^{\mathcal{D}_N}$, we have

$$\operatorname{ord}_s(\eta^X; \Gamma_0(N)) = \frac{1}{24} \sum_{d \in \mathcal{D}_N} \frac{N \cdot \gcd(d, t)^2}{d \cdot \gcd(t^2, N)} X_d\,. \tag{5.6}$$

In particular, that implies

$$\operatorname{ord}_{a/t}(\eta^X\,;\Gamma_0(N)) = \operatorname{ord}_{1/t}(\eta^X\,;\Gamma_0(N)) \tag{5.7}$$

for all $t \in \mathcal{D}_N$ and for all the $\varphi(\gcd(t, N/t))$ inequivalent cusps of $\Gamma_0(N)$ represented by rational numbers of the form $\dfrac{a}{t} \in \mathcal{S}_N$ with $\gcd(a, t) = 1$, where φ denotes Euler's totient function.

We define the *order map* $\mathcal{O}_N : \mathbb{Z}^{\mathcal{D}_N} \to \frac{1}{24}\mathbb{Z}^{\mathcal{D}_N}$ of level N as the map which sends $X \in \mathbb{Z}^{\mathcal{D}_N}$ to the ordered set of orders of the eta quotient η^X at the cusps $\{1/t\}_{t\in\mathcal{D}_N}$ of $\Gamma_0(N)$. Also, we define *order matrix* $A_N \in \mathbb{Z}^{\mathcal{D}_N\times\mathcal{D}_N}$ of level N by

$$A_N(t, d) := 24 \cdot \operatorname{ord}_{1/t}(\eta_d\,;\Gamma_0(N)) \tag{5.8}$$

for all $t, d \in \mathcal{D}_N$. By linearity of the order map, we have

$$\mathcal{O}_N(X) = \frac{1}{24} \cdot A_N X\,. \tag{5.9}$$

From (5.8) and (5.5), we note that the matrix A_N is not symmetric. It would have been much easier for us to work with A_N if it would have been symmetric (for example, see (6.1)). So, we define the *symmetrized order matrix* $\widehat{A}_N \in \mathbb{Z}^{\mathcal{D}_N\times\mathcal{D}_N}$ by

$$\widehat{A}_N(t, \_) = \gcd(t, N/t) \cdot A_N(t, \_) \quad \text{for all } t \in \mathcal{D}_N, \tag{5.10}$$

where $\widehat{A}_N(t, \_)$ (resp. $A_N(t, \_)$) denotes the row of $\widehat{A}_N$ (resp. A_N) indexed by $t \in \mathcal{D}_N$. For example, for a prime power p^n, we have

$$\widehat{A}_{p^n} = \begin{pmatrix} p^n & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-1} & p^n & p^{n-1} & \cdots & p^2 & p \\ p^{n-2} & p^{n-1} & p^n & \cdots & p^3 & p^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ p & p^2 & p^3 & \cdots & p^n & p^{n-1} \\ 1 & p & p^2 & \cdots & p^{n-1} & p^n \end{pmatrix}. \tag{5.11}$$

For $r \in \mathbb{N}$, if $Y, Y' \in \mathbb{Z}^{\mathcal{D}_N^r}$ is such that $Y - Y'$ is nonnegative (resp. positive) at each element of $\mathcal{D}_N^r$, then we write $Y \geq Y'$ (resp. $Y > Y'$). In particular, for $X \in \mathbb{Z}^{\mathcal{D}_N}$, the eta quotient η^X is holomorphic if and only if $\widehat{A}_N X \geq 0$. From (5.10), (5.8) and (5.5), we note that $\widehat{A}_N(t, d)$ is multiplicative in N and in $d, t \in \mathcal{D}_N$. Hence, it follows that

$$\widehat{A}_N = \bigotimes_{\substack{p^n \| N \\ p \text{ prime}}} \widehat{A}_{p^n}, \tag{5.12}$$

where by $\otimes$, we denote the Kronecker product of matrices.*

*Kronecker product of matrices is not commutative. However, since any given ordering of the primes dividing N induces a lexicographic ordering on $\mathcal{D}_N$ with which the entries of $\widehat{A}_N$ are indexed, Equation (5.12) makes sense for all possible orderings of the primes dividing N.

It is easy to verify that for a prime power p^n, the matrix $\widehat{A}_{p^n}$ is invertible with the tridiagonal inverse:

$$\widehat{A}_{p^n}^{-1} = \frac{1}{p^n(1-\frac{1}{p^2})}\begin{pmatrix} 1 & -\frac{1}{p} & & & & \\ -\frac{1}{p} & 1+\frac{1}{p^2} & -\frac{1}{p} & & \text{\Large 0} & \\ & -\frac{1}{p} & 1+\frac{1}{p^2} & -\frac{1}{p} & & \\ & & \ddots & \ddots & \ddots & \\ & \text{\Large 0} & & -\frac{1}{p} & 1+\frac{1}{p^2} & -\frac{1}{p} \\ & & & & -\frac{1}{p} & 1 \end{pmatrix}. \tag{5.13}$$

For general N, the invertibility of the matrix $\widehat{A}_N$ now follows by (5.12). Hence, any eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders at the set of the cusps $\{1/t\}_{t\in\mathcal{D}_N}$ of $\Gamma_0(N)$. In particular, for distinct $X, X' \in \mathbb{Z}^{\mathcal{D}_N}$, we have $\eta^X \neq \eta^{X'}$. The last statement is also implied by the uniqueness of q-series expansion: Let $\eta^{\widehat{X}}$ and $\eta^{\widehat{X}'}$ be the *eta products* (i. e. $\widehat{X}, \widehat{X}' \geq 0$) obtained by multiplying η^X and $\eta^{X'}$ with a common denominator. The claim follows by induction on the weight of $\eta^{\widehat{X}}$ (or equivalently, the weight of $\eta^{\widehat{X}'}$) when we compare the corresponding first two exponents of q occurring in the q-series expansions of $\eta^{\widehat{X}}$ and $\eta^{\widehat{X}'}$.

6. A holomorphy preserving map

For $d \in \mathcal{D}_N$, we say that d *exactly divides* N (and write $d\|N$) if d and N/d are mutually coprime. For any $d\|N$, there exists a canonical bijection between $\mathbb{Z}^{\mathcal{D}_N}$ and $\mathbb{Z}^{\mathcal{D}_{N/d}\times\mathcal{D}_d}$. We denote by $X^{[d]}$ the image of $X \in \mathbb{Z}^{\mathcal{D}_N}$ in $\mathbb{Z}^{\mathcal{D}_{N/d}\times\mathcal{D}_d}$ under this bijection. From the facts that $\widehat{A}_N = \widehat{A}_d \otimes \widehat{A}_{N/d}$ and that these matrices are symmetric, it follows that

$$(\widehat{A}_N X)^{[d]} = \widehat{A}_{N/d} X^{[d]} \widehat{A}_d \tag{6.1}$$

(see Lemma 4.3.1 in [9]).

Now we provide a holomorphy preserving map which will be very useful in the next two sections. Let $M, N \in \mathbb{N}$ with $N\|M$ and let $\{a_n\}_{n\in\mathbb{N}}$ be a multiplicative sequence of integers such that $a_n = 0$ if $n \notin \mathcal{D}_{M/N}$ and

$$a_{p^{j-1}} + a_{p^{j+1}} \leq \begin{cases} p a_{p^j} & \text{if } p^j \| (M/N) \\ (p+\frac{1}{p}) a_{p^j} & \text{otherwise} \end{cases} \tag{6.2}$$

for all primes p and for all nonnegative integers j such that $p^j \,|\, (M/N)$. Here, we set $a_{p^{-1}} := 0$ for each prime p. We define $\widehat{a} \in \mathbb{Z}^{\mathcal{D}_{M/N}}$ by $\widehat{a}_d := a_d$ for all $d \in \mathcal{D}_{M/N}$.

Lemma 1. *Let M, N, $\{a_n\}_{n\in\mathbb{N}}$ and $\widehat{a}$ be as above. Then the homomorphism*

$$\varPhi_{M,N,\widehat{a}}(\eta^X) := \eta^{X^{[M/N]}\cdot\widehat{a}} \tag{6.3}$$

from the multiplicative group of eta quotients on $\Gamma_0(M)$ to that on $\Gamma_0(N)$ preserves holomorphy. Moreover, if $\widehat{a}$ is such that the strict inequality in (6.2) holds for all prime powers p^j which divide M/N, then $\varPhi_{M,N,\widehat{a}}$ does not map any nonconstant holomorphic eta quotient to 1.

Proof. Let $X \in \mathbb{Z}^{\mathcal{D}_M}$ be such that the eta quotient η^X is holomorphic. Since η^X is holomorphic if and only if

$$\widehat{A}_M X \geq 0, \tag{6.4}$$

we only require to show that

$$\widehat{A}_N X^{[M/N]} \widehat{a} \geq 0. \tag{6.5}$$

From (6.1), we get:

$$(\widehat{A}_M X)^{[M/N]} \widehat{A}_{M/N}^{-1} \widehat{a} = \widehat{A}_N X^{[M/N]} \widehat{a}, \tag{6.6}$$

It follows from the last equality and (6.4) that it is enough to show:

$$\widehat{A}_{M/N}^{-1} \widehat{a} \geq 0. \tag{6.7}$$

From multiplicativity of the sequence $\{a_n\}_{n\in\mathbb{N}}$, we get that

$$\widehat{a} = \bigotimes_{\substack{p^n \| (M/N) \\ p \text{ prime}}} \widehat{a}^{(p^n)}, \tag{6.8}$$

where $\widehat{a}^{(p^n)} \in \mathbb{Z}^{\mathcal{D}}_{p^n}$ is defined by $\widehat{a}^{(p^n)}_{p^j} := a_{p^j}$ for all j such that $p^j | (M/N)$. Now, (5.12) and (6.8) together imply that for (6.7) to hold, it suffices if so does the following inequality:

$$\widehat{A}_{p^n}^{-1} \widehat{a}^{(p^n)} \geq 0 \tag{6.9}$$

for each prime power $p^n \| (M/N)$. The last inequality follows immediately from (5.13) and (6.2).

The strict inequality in (6.2) implies the strict inequality in (6.9) which in turn, implies the strict inequality in (6.7). Since η^X is holomorphic, it has nonnegative order of vanishing at all cusps of $\Gamma_0(M)$. Moreover, if η^X is nonconstant, then its order of vanishing at some cusp of $\Gamma_0(M)$ is nonzero. So, (5.9) implies that all the entries of $A_M X$ are nonnegative and at least one of its entries is nonzero. Hence via (6.6), we conclude that $\widehat{A}_N X^{[M/N]} \widehat{a} \neq 0$. □

Corollary 2. *Let $M \in \mathbb{N}$ and let p be a prime divisor of M. Let $M = p^n N$, where $p \nmid N$. For some $j \in \{0, 1, \ldots, n\}$ and for some $m \in \{1, \ldots, p-1\}$, define $\widehat{a} = \widehat{a}_{(j,m)} \in \mathbb{Z}^{\mathcal{D}_{p^n}}$ by*

$$\widehat{a}(p^i) = \begin{cases} m & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases} \tag{6.10}$$

Then the homomorphism $\Phi_{M,N,\widehat{a}}$ preserves holomorphy (see (6.3)) and it does not map any nonconstant holomorphic eta quotient to 1.

Corollary 3. *Let M, N and $p \geq 3$ be as in Corollary 2. Define $\mathbb{1}_N \in \mathbb{Z}^{\mathcal{D}_N}$ by*

$$\mathbb{1}_N(d) := 1 \text{ for all } d \in \mathcal{D}_N, \tag{6.11}$$

Let $X \in \mathbb{Z}^{\mathcal{D}_M}$ be such that η^X is a holomorphic eta quotient of weight $1/2$ on $\Gamma_0(M)$. Then $\Phi_{M,p^n,\mathbb{1}_N}(\eta^X) = \eta_{p^{j_0}}$ for some $j_0 \in \{0, 1, \ldots, n\}$.

Proof. Let $j \leq n$ be a nonnegative integer and define $\widehat{a} \in \mathbb{Z}^{\mathcal{D}_{p^n}}$ by

$$\widehat{a}(p^i) = \begin{cases} p-1 & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases} \tag{6.12}$$

Let $Y \in \mathbb{Z}^{\mathcal{D}_N}$ be such that $\eta^Y = \varPhi_{M,N,\widehat{a}}(\eta^X)$. Then Corollary 2 implies that η^Y is holomorphic. Let $Z \in \mathbb{Z}^{\mathcal{D}_{p^n}}$ be such that $\eta^Z = \varPhi_{M,p^n,\mathbb{1}_N}(\eta^X)$. Then from (5.2), (6.3) and (6.12), it follows that

$$\sigma(Y) = \sigma(X) + (p-2)Z_{p^j} = 1 + (p-2)Z_{p^j}, \tag{6.13}$$

where the last equality holds since η^X is of weight $1/2$. Since $p \geq 3$, we have $\sigma(Y) \geq 0$ if and only if $Z_{p^j} \geq 0$. Since η^Y is holomorphic, $\sigma(Y)$ is indeed nonnegative and so is Z_{p^j} for all $j \in \{0, 1, \dots, n\}$. Since the map $\varPhi_{M,p^n,\mathbb{1}_N}$ preserves weight, we have

$$\sigma(Z) = \sigma(X) = 1.$$

Hence, there exists $j_0 \in \{0, 1, \dots, n\}$ such that

$$Z_{p^j} = \begin{cases} 1 & \text{if } j = j_0 \\ 0 & \text{otherwise.} \end{cases} \tag{6.14}$$

In other words, we have $\eta^Z = \eta_{p^{j_0}}$. □

In particular, if we set $M = p^n$ in Corollary 3, then $\varPhi_{M,p^n,\mathbb{1}_N}$ becomes the identity map on eta quotients on $\Gamma_0(M)$. So, Corollary 3 implies that

Corollary 4. *For all primes $p \geq 3$ and for all $n \in \mathbb{N}$, the only holomorphic eta quotient of weight $1/2$ and level p^n is η_{p^n}.*

7. Reduction to 3-smooth levels

For $m \in \mathbb{N}$, an integer N is called m-smooth if none of the prime factors of N is greater than m. We have

Lemma 2. *If for all 3-smooth $N \in \mathbb{N}$, the only primitive holomorphic eta quotients on $\Gamma_0(N)$ are those given in Zagier's list (see Theorem 1), then the same is true for all $N \in \mathbb{N}$.*

Proof. We shall proceed by induction on the greatest prime divisor of N: Let us assume that for some $m \geq 3$, the only primitive holomorphic eta quotients on $\Gamma_0(N)$ for each m-smooth $N \in \mathbb{N}$ are those given in Zagier's list.

Suppose, there exists a primitive holomorphic eta quotient of some level $M = p^n N$ with $p > m$, where N is m-smooth. Let $Y \in \mathbb{Z}^{\mathcal{D}_{p^n}}$ be such that $\eta^Y = \varPhi_{M,p^n,\mathbb{1}_N}(\eta^X)$. From Corollary 3, we know that there exists a $j_0 \in \{0, 1, \dots, n\}$ such that $\eta^Y = \eta_{p^{j_0}}$. Let $i_0 \in \{0, n\}$ be distinct from j_0. Then we have $Y_{p^{i_0}} = 0$. Define $\widehat{a} \in \mathbb{Z}^{\mathcal{D}_{p^n}}$ by

$$\widehat{a}(p^i) = \begin{cases} 4 & \text{if } i = i_0 \\ 1 & \text{otherwise.} \end{cases} \tag{7.1}$$

Since $p \geq 5$, Corollary 2 implies that both $\varPhi_{M,N,\widehat{a}}(\eta^X)$ and $\varPhi_{M,N,\mathbb{1}_{p^n}}(\eta^X)$ are holomorphic. Let $Z \in \mathbb{Z}^{\mathcal{D}_N}$ such that $\eta^Z = \varPhi_{M,N,\widehat{a}}(\eta^X)$. Then from

(5.2), (6.3) and (7.1), it follows that

$$\sigma(Z) = \sigma(X) + 3Y_{p^{i_0}} = 1, \tag{7.2}$$

where the last equality holds since η^X is of weight $1/2$ and since $Y_{p^{i_0}} = 0$. Hence, the eta quotient $f := \eta^Z$ is of weight $1/2$. Again, since the map $\Phi_{M,N,\mathbb{1}_{p^n}}$ preserves weight, the eta quotient $g := \Phi_{M,N,\mathbb{1}_{p^n}}(\eta^X)$ is also of weight $1/2$. Since η^X is a primitive eta quotient of level M, there exists some $d|N$ such that $X^{[p^n]}_{d,p^{i_0}} \neq 0$. So from (6.3) and (7.1), it follows that $\Phi_{M,N,\widehat{a}}(\eta^X) \neq \Phi_{M,N,\mathbb{1}_{p^n}}(\eta^X)$, i. e. $f \neq g$. Now, by the induction hypothesis, we have: $f = f'_{d_1}$ and $g = g'_{d_2}$, where f' and g' belong to Zagier's list and $d_1, d_2 \in \mathbb{N}$. Here, f'_{d_1} (resp. g'_{d_2}) denotes the rescaling of f' by d_1 (resp. the rescaling of g' by d_2). From (7.1), we see that the corresponding exponents of f and g are congruent modulo 3. Hence from Zagier's list , it follows that $d_1 = d_2$ and that the set $\{f', g'\}$ is either $\left\{\dfrac{\eta^2}{\eta_2}, \dfrac{\eta_2^2}{\eta}\right\}$ or $\left\{\dfrac{\eta_2^5}{\eta^2\eta_4^2}, \dfrac{\eta\eta_4}{\eta_2}\right\}$. Both of these possibilities imply that the eta quotient $\Phi_{M,N,\widehat{a}'}(\eta^X)$ is not holomorphic, where $\widehat{a}' \in \mathbb{Z}^{\mathcal{D}_{p^n}}$ is defined by

$$\widehat{a}'(p^i) = \begin{cases} 2 & \text{if } i = i_0 \\ 1 & \text{otherwise.} \end{cases} \tag{7.3}$$

Thus, we get a contradiction to Corollary 2! □

8. The completeness of Zagier's list

In the following, we shall show that each holomorphic eta quotient of weight $1/2$ and of a 3-smooth level is a rescaling of some eta quotient on $\Gamma_0(72)$ by a positive integer. By using standard linear algebraic techniques (for example, see Chapter 4 in [10]), it is easy to check that the only primitive holomorphic eta quotients of weight $1/2$ on $\Gamma_0(72)$ are those given in Zagier's list. Thus, we obtain the completeness of the list.

Lemma 3. *For an integer $n > 2$ and for $X \in \mathbb{Z}^{D_{2^n}}$, if the eta quotient η^X is holomorphic, then we have*

$$|X_1| + |X_{2^n}| \leq 2 \cdot \sigma(X). \tag{8.1}$$

Moreover, if equality holds in (8.1), then both X_1 and X_{2^n} are even.

Proof. Let $X \in \mathbb{Z}^{\mathcal{D}_{2^n}}$ and define $\widehat{a} \in \mathbb{Z}^{\mathcal{D}_{2^n}}$ by

$$\widehat{a}(2^j) = \begin{cases} 2 - \operatorname{sgn}(X_{2^j}) & \text{if } j = 0 \text{ or } j = n. \\ 2 & \text{otherwise.} \end{cases} \tag{8.2}$$

Then from Lemma 1, it follows that $\Phi_{2^n,1,\widehat{a}}(\eta^X) = \eta^{2\cdot\sigma(X)-|X_1|-|X_{2^n}|}$ is holomorphic. So, we have

$$|X_1| + |X_{2^n}| \leq 2 \cdot \sigma(X).$$

Let $Y := \widehat{A}_{2^n}X$. Since η^X is holomorphic, $Y \geq 0$. Now, if equality holds in (8.1), then $\Phi_{2^n,1,\widehat{a}}(\eta^X) = \eta^{X^{\mathrm{T}}\widehat{a}} = 1$. In other words, we have

$$Y^{\mathrm{T}}\widehat{A}_{2^n}^{-1}\widehat{a} = 0, \tag{8.3}$$

since $\widehat{A}_{2^n}$ is symmetric. From (5.13) and (8.2), we get that

$$\widehat{A}_{2^n}^{-1}\widehat{a} = \frac{1}{3\cdot 2^{n-1}}\bigl(2(1-\operatorname{sgn}(X_1)),\, 1+\operatorname{sgn}(X_1),\, 1, 1, \ldots \\ \ldots, 1, 1+\operatorname{sgn}(X_{2^n}),\, 2(1-\operatorname{sgn}(X_{2^n}))\bigr)^{\mathrm{T}}. \tag{8.4}$$

From (8.3) and (8.4), it follows that $Y_{2^j} = 0$ for all j except at most two nonconsecutive values (say, j_1 and j_2) of $j \in \{0, 1, n-1, n\}$. That implies

$$X = Y_{2^{j_1}} \cdot \widehat{A}_{2^n}^{-1}(\_, 2^{j_1}) + Y_{2^{j_2}} \cdot \widehat{A}_{2^n}^{-1}(\_, 2^{j_2}), \tag{8.5}$$

where $\widehat{A}_{2^n}(\_, 2^j)$ denotes the column of $\widehat{A}_{2^n}$ indexed by 2^j. From (8.5) and from (5.13), it follows that the values of X at the elements of $\mathcal{D}_{2^n}$ are integral if and only if both X_1 and X_{2^n} are even. $\square$

In particular, it follows trivially from the above lemma that

Corollary 5. *Every holomorphic eta quotient of weight $1/2$ and level 2^n is a rescaling of some holomorphic eta quotient on $\Gamma_0(4)$.*

Lemma 4. *If there exists a primitive holomorphic eta quotient of weight $1/2$ and level $N = 2^m 3^n$, then $m \le 3$ and $n \le 2$.*

Proof. Suppose $X \in \mathbb{Z}^{\mathcal{D}_N}$ is such that η^X is a primitive holomorphic eta quotient of weight $1/2$ and level N.

Suppose $n \ge 3$. It follows from Corollary 3 that $\varPhi_{N,3^n,\mathbb{1}_{2^m}}(\eta^X) = \eta_{3^{j_0}}$ for some $j_0 \in \{0, 1, \ldots, n\}$. Define $(i_1, i_2) \in \mathbb{Z}^2$ by

$$(i_1, i_2) := \begin{cases} (0,1) & \text{if } n \le 2j_0 \\ (n, n-1) & \text{otherwise} \end{cases} \tag{8.6}$$

and define $\widehat{a}, \widehat{a}' \in \mathbb{Z}^{\mathcal{D}_{3^n}}$ by

$$\widehat{a}(3^i) = \begin{cases} 5 & \text{if } i = i_1 \\ 2 & \text{if } i = i_2 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \widehat{a}'(3^i) = \begin{cases} 2 & \text{if } i = i_2 \\ 1 & \text{otherwise.} \end{cases} \tag{8.7}$$

Then from Lemma 1, it follows that both $f := \varPhi_{N,2^m,\widehat{a}}(\eta^X)$ and $g := \varPhi_{N,2^m,\widehat{a}'}(\eta^X)$ are holomorphic. Also, it follows from our choice of i_1 and i_2 that both f and g are of weight $1/2$. Since η^X is a primitive eta quotient of level N, there exists some $j \in \{0, 1, \ldots, m\}$ such that $X^{[3^n]}_{2^j,3^{i_1}} \neq 0$. So from (6.3) and (8.7), it follows that $\varPhi_{N,2^m,\widehat{a}}(\eta^X) \neq \varPhi_{N,2^m,\widehat{a}'}(\eta^X)$, i. e. $f \neq g$. Corollary 5 implies that $f = f'_{d_1}$ and $g = g'_{d_2}$, where f' and g' are two primitive holomorphic quotients of weight $1/2$ on $\Gamma_0(4)$ and $d_1, d_2 \in \mathbb{N}$. Here, f'_{d_1} (resp. g'_{d_2}) denotes the rescaling of f' (resp. the rescaling of g') by d_1 (resp. d_2). It is easy to check the following fact:

(8.8) *The set of primitive holomorphic eta quotients of weight $1/2$ on $\Gamma_0(4)$ is a subset of Zagier's list.*

From (8.7), we see that the corresponding exponents of f and g are congruent modulo 4. Since none of the exponents in the eta quotients in Zagier's list is a multiple of 4, it follows that $d_1 = d_2$. So, the corresponding exponents of f' and g' are congruent modulo 4. But again from Zagier's list, we see that there is no pair of primitive holomorphic eta quotients of weight $1/2$ on $\Gamma_0(4)$ whose corresponding exponents are congruent modulo 4. Thus, we get a contradiction! Hence, we conclude that $n \leq 2$.

Suppose $m \geq 4$. For $j \in \{0, 1, \dots, n\}$, define $\delta_j \in \mathbb{Z}^{\mathcal{D}_{3^n}}$ by

$$\delta_j(3^i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{8.9}$$

It follows from Corollary 2 that for all $j \in \{0, 1, \dots, n\}$ and for all $r \in \{0, 1\}$, the eta quotients

$$f_{(j,r)} := \Phi_{N, 2^m, \mathbb{1}_{3^n} + r\delta_j}(\eta^X) \tag{8.10}$$

are holomorphic. Let $Y \in \mathbb{Z}^{\mathcal{D}_{3^n}}$ be such that $\eta^Y = \Phi_{N, 3^n, \mathbb{1}_{2^m}}(\eta^X)$. It is easy to see that the weight of $f_{(j,r)}$ is $(1 + Y_j)/2$. From Corollary 3, we know that there exists a $j_0 \in \{0, 1, \dots, n\}$ such that $\eta^Y = \eta_{3^{j_0}}$. So, $f_{(j,r)}$ is a holomorphic eta quotient of weight $1/2$ on $\Gamma_0(2^m)$ if and only if $(j, r) \neq (j_0, 1)$. Let $g := f_{(0,0)}$. Corollary 5 implies in particular, that each holomorphic eta quotient of weight $1/2$ on $\Gamma_0(2^m)$ is either nonprimitive or not of level 2^m.

First consider the case where g is neither primitive nor of level 2^m. Then for $j \neq j_0$, each of the eta quotients $g_{(j)} := f_{(j,1)}/g$ is either nonprimitive or not of level 2^m. Now, if for all $j \neq j_0$, the eta quotient $g_{(j)}$ is nonprimitive (resp. not of level 2^m), then η^X is nonprimitive (resp. not of level 2^m) which is contrary to our assumption! In particular, that is the case if $n < 2$. So, $n = 2$. Moreover, there are distinct $j_1, j_2 \in \{0, 1, 2\} \smallsetminus \{j_0\}$ such that the eta quotient $g_{(j_1)}$ is nonprimitive and of level 2^m, whereas the eta quotient $g_{(j_2)}$ is primitive but not of level 2^m. Since both $f_{(j_1,1)}$ and $f_{(j_2,1)}$ are of weight $1/2$, it follows from Lemma 3 that

$$f_{(j_1,1)} = h_2 \eta_{2^m}^{\pm\epsilon_1}, \quad f_{(j_2,1)} = \eta^{\pm\epsilon_2} h_2' \quad \text{and} \quad g_{(j_0)} = \eta^{\mp\epsilon_2} h_2'' \eta_{2^m}^{\mp\epsilon_1}, \tag{8.11}$$

where h_2 (resp. h_2', h_2'') denotes the rescaling of an eta quotient h (resp. h', h'') on $\Gamma_0(2^{m-2})$ by 2 and $\epsilon_1, \epsilon_2 \in \{1, 2\}$. The last equality in (8.11) follows, because in the case which we are presently considering, the eta quotient $g = g_{(j_0)} f_{(j_1,1)} f_{(j_2,1)}$ is neither primitive nor of level 2^m. Let $\ell \in \{1, 2\}$ be such that $|j_0 - j_\ell| = 1$. It follows from Lemma 1 that the eta quotient

$$\alpha := \Phi_{N, 2^m, \mathbb{1}_{3^n} + \delta_{j_0} + 4\delta_{j_\ell}}(\eta^X) \tag{8.12}$$

is holomorphic. It is easy to see that the weight of α is $(1 + Y_{j_0} + 4Y_{i_0})/2 = 1$. Since $\alpha = f_{(j_\ell,1)}^4 \cdot g_{(j_0)}$ is a holomorphic eta quotient of weight 1 and level 2^m, Lemma 3 implies that

$$3|\epsilon_\ell| + |\epsilon_{3-\ell}| \leq 4. \tag{8.13}$$

The only solution to the above inequality in $\epsilon_\ell, \epsilon_{3-\ell} \in \{1, 2\}$ is $\epsilon_\ell = \epsilon_{3-\ell} = 1$, which contradicts Lemma 3! Hence, either the eta quotient g is nonprimitive and of level 2^m or g is primitive but not of level 2^m.

Let $Z \in \mathbb{Z}^{\mathcal{D}_{2^m}}$ be such that $\eta^Z = g$. Then there exists a unique $r \in \{0, m\}$ such that $Z_{2^r} = 0$. Let $s, t \in \{0, 1, \dots, m\}$ be such that $|r - s| = 1$ and $t = m - r$. Then Z_{2^t} is nonzero. Since $m \geq 4$, we have $|s - t| \geq 3$. Since

Z_{2^t} is nonzero, it follows from Corollary 5 and (8.8) that $Z_{2^s} = 0$. For $j \in \{0, 1, \ldots, m\}$, define $\delta'_j \in \mathbb{Z}^{\mathcal{D}_{2^m}}$ by

$$\delta'_j(2^i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{8.14}$$

Define $\widehat{b}, \widehat{b}' \in \mathbb{Z}^{\mathcal{D}_{2^m}}$ by

$$\widehat{b} := (1 + |Z_{2^t}|)\mathbb{1}_{2^m} + \delta'_r + \delta'_s - \operatorname{sgn}(Z_{2^t})\delta'_t, \tag{8.15}$$

$$\widehat{b}' := (1 + |Z_{2^t}|)\mathbb{1}_{2^m} + 3\delta'_r + \delta'_s - \operatorname{sgn}(Z_{2^t})\delta'_t. \tag{8.16}$$

Then from Lemma 1, it follows that both of the eta quotients $\beta := \Phi_{N,3^n,\widehat{b}}(\eta^X)$ and $\gamma := \Phi_{N,3^n,\widehat{b}'}(\eta^X)$ are holomorphic. Also, it follows from our choice of r, s and t that both β and γ are of weight $1/2$. Since η^X is a primitive eta quotient of level N, there exists some $j \in \{0, 1, \ldots, n\}$ such that $X^{[2^m]}_{2^r,3^j} \neq 0$. So from (6.3), (8.15) and (8.16), it follows that $\Phi_{N,3^n,\widehat{b}}(\eta^X) \neq \Phi_{N,3^n,\widehat{b}'}(\eta^X)$, i. e. $\beta \neq \gamma$. From (8.15) and (8.16), we see that the corresponding exponents of β and γ are congruent modulo 2. Thus, we have obtained two distinct holomorphic eta quotients of weight $1/2$ on $\Gamma_0(3^n)$ whose corresponding exponents are congruent modulo 2. This contradicts Corollary 4! Hence, we conclude that $m \leq 3$. □

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